ON MACKEY CONVERGENCE IN LOCALLY CONVEX SPACES

BY

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ABSTRACT

After some introductory propositions, we give a dual characterization of those locally convex spaces which satisfy the Mackey convergence condition or the fast convergence condition by means of Schwartz topologies. Making use of the universal Schwartz space $(l_\infty, \tau(l_\infty, l_1))$ we prove some representation theorems for bornological and ultrabornological spaces, that is, every bornological space E is a dense subspace of an inductive limit lim ind E_a, $a \in A$, of *separable* Banach spaces E_a, and every Mackey null sequence in E is a null sequence in some E_a. If E is ultrabornological, then E can be represented as lim ind E_a, $a \in A$, all E_a separable Banach spaces, such that every fast null sequence in E is a null sequence in some E_a.

1. Introduction

The present paper is devoted to the concept of Mackey convergence in locally convex spaces and especially in bornological spaces (refer to [5, Sec. 28.3]). Among other things we give a complete characterization of those locally complete bornological spaces which satisfy Mackey's convergence condition. This result is based (i) on the dual characterization of Mackey's convergence condition for arbitrary locally convex spaces by means of Schwartz topologies as stated in Proposition 9, and (ii) on the constructive proofs of the existence of universal Schwartz spaces given independently in the recent papers [4] and [6].

2. Notation

Let E be a locally convex space (always tacitly assumed to be Hausdorff). Every closed, bounded, and absolutely convex subset B of E is called a *disk* (in E).

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If B is a disk, we denote by E_B the linear subspace $\bigcup_{n=1}^{\infty} n \cdot B$ of E, normed by the gauge of B in E_B. The disk B is called a *Banach disk* if E_B is even complete. E is called *locally complete*, if its system of bounded subsets has a fundamental system consisting of Banach disks.

A sequence $(x_n)_{n \in \mathbb{N}}$ in E is called a *Mackey* (or *local*) *null sequence* if it is a null sequence in some E_B, for $B \subset E$ a disk. Obviously, every Mackey null sequence is a null sequence in E. If the converse is also true we say that E satisfies *Mackey's convergence condition.*

For every absolutely convex and absorbing subset U of E we denote by $E_{(U)}$ the factor space $E/N(U)$, normed by $\| x + N(U) \| = p_U(x)$. Here $N(U)$ is the kernel of the gauge p_U of U.

In the dual E' of E we will consider various $\mathfrak{S}$-topologies, $\mathfrak{S}$ being a saturated family of bounded sets in E which we always will assume to be a covering of E. We call $\mathfrak{S}$ a *Schwartz family* if for every $A \in \mathfrak{S}$ there exists a disk $B \in \mathfrak{S}$ containing A such that A is precompact in E_B. Of course we may also assume that A is a disk. $E'_{(A_0)}$ and $E'_{(B_0)}$ are subspaces of $(E_A)'$ and of $(E_B)'$, respectively, and the adjoint of the natural injection $E_A \to E_B$ induces a precompact linear map of $E'_{(B_0)}$ onto $E'_{(A_0)}$ which is nothing else but the canonical projection. Conversely, if this map is precompact, then its adjoint $(E'_{(A_0)})' \to (E'_{(B_0)})'$ is compact and induces the original injection $E_A \to E_B$ between the respective subspaces E_A, E_B, which is precompact. So we have proved Proposition 1 which generalizes results given in [3] and [7].

PROPOSITION 1. *$\mathfrak{S}$ is a Schwartz family if and only if the corresponding $\mathfrak{S}$-topology $\mathfrak{T}_{\mathfrak{S}}$ on E' is a Schwartz topology.*

For details on Schwartz spaces which we cite often without special reference, the reader may consult [2] and [7].

A sequence $(x_n)_{n \in \mathbb{N}}$ in E is called an *$\mathfrak{S}$-null sequence* if there exists a sequence $(r_n)_{n \in \mathbb{N}}$ of positive real numbers tending to infinity such that $\{r_n x_n \mid n \in \mathbb{N}\} \in \mathfrak{S}$. Exactly as in [5, Sec. 28.3, (1)] one proves Proposition 2.

PROPOSITION 2. *$(x_n)_{n \in \mathbb{N}}$ is an $\mathfrak{S}$-null sequence in E if and only if it is a null sequence in some E_B, $B \in \mathfrak{S}$ a disk.*

If $\mathscr{B}$ is the family of all bounded subsets in E, then the $\mathscr{B}$-null sequences are just the Mackey null sequences in E.

3. Associated Schwartz families

Let us begin with a generalization of some of the results of Hogbé-Nlend [1] which are given without proof. It is clear that our previous statements have easy reformulations within the theory of bornologies.

Let $\mathfrak{S}$ be any saturated family of bounded subsets in a given locally convex space E. Let us denote by $\mathfrak{S}_0$ the saturated hull of all $\mathfrak{S}$-null sequences. Then we have $\mathfrak{S}_0 \subset \mathfrak{S}$, and a fundamental system in $\mathfrak{S}_0$ is given by the $\sigma(E, E')$-closed absolutely convex covers of the $\mathfrak{S}$-null sequences.

First we prove Proposition 3.

PROPOSITION 3. *Every $\mathfrak{S}$-null sequence is an $\mathfrak{S}_0$-null sequence, and conversely.*

PROOF. From Proposition 2 and from $\mathfrak{S}_0 \subset \mathfrak{S}$ follows that every $\mathfrak{S}_0$-null sequence is an $\mathfrak{S}$-null sequence. On the other hand, if $(x_n)_{n \in \mathbb{N}}$ is an $\mathfrak{S}$-null sequence we may choose a sequence $r_n \to \infty$ of positive real numbers such that $\{r_n x_n \mid n \in \mathbb{N}\} \in \mathfrak{S}$. Since $(r_n^{\frac{1}{2}} \cdot x_n)_{n \in \mathbb{N}}$ is an $\mathfrak{S}$-null sequence, it follows from $r_n^{\frac{1}{2}} \to \infty$ and $\{r_n^{\frac{1}{2}} \cdot x_n \mid n \in \mathbb{N}\} \in \mathfrak{S}_0$ that $(x_n)_{n \in \mathbb{N}}$ is an $\mathfrak{S}_0$-null sequence.

We now state a corollary of the proposition.

COROLLARY 4. $\mathfrak{S}_{00} = \mathfrak{S}_0$.

Next we prove Proposition 5.

PROPOSITION 5. *$\mathfrak{S}_0$ is a Schwartz family.*

PROOF. Every $\mathfrak{S}_0$-null sequence $(x_n)_{n \in \mathbb{N}}$ is a null sequence in some E_B, $B \in \mathfrak{S}_0$ a disk. Hence the absolutely convex cover M of $\{x_n \mid n \in \mathbb{N}\}$ is precompact in E_B. Since B is closed in E, the closures of M in E and in E_B coincide; refer to [5, Sec. 28.5, (2), (3), (4)]. From this it follows that every $A \in \mathfrak{S}_0$ is contained and precompact in some E_B, $B \in \mathfrak{S}_0$ a disk.

We call $\mathfrak{S}_0$ the *Schwartz family associated with* $\mathfrak{S}$.

PROPOSITION 6. *$\mathfrak{S}$ is a Schwartz family if and only if $\mathfrak{S} = \mathfrak{S}_0$.*

PROOF. If $\mathfrak{S} = \mathfrak{S}_0$, then $\mathfrak{S}$ is a Schwartz family by Proposition 5. Now let $\mathfrak{S}$ be a Schwartz family. If A is in $\mathfrak{S}$ then there is a disk $B \in \mathfrak{S}$ such that A is precompact in E_B. By a corollary of the Banach-Dieudonné theorem (refer to [5, Sec. 21.10, (3)]) A is contained in the E_B-closure of the absolutely convex cover M of some null sequence in E_B. Hence A is in the E-closure $\bar{M} \in \mathfrak{S}_0$ of M.

If $\mathfrak{S}$ and $\mathfrak{S}'$ are saturated families of bounded sets in E, we have $\mathfrak{S}'_0 \subset \mathfrak{S}_0$ if

$\mathfrak{S}' \subset \mathfrak{S}$. From this we can state Proposition 7 with the aid of Proposition 6.

PROPOSITION 7. *The $\mathfrak{S}_0$-topology is the finest among all Schwartz $\mathfrak{S}'$-topologies τ on E' which are coarser than the $\mathfrak{S}$-topology.*

Now we turn our interest to the family $\mathscr{B}$ of *all* bounded subsets of E. We will denote the corresponding $\mathscr{B}_0$-topology on E' by $\tau_{c_0}(E'\ E)$. This is the topology of uniform convergence on all the Mackey null sequences of E. From Proposition 7 we can state the following proposition.

PROPOSITION 8. *$\tau_{c_0}(E'\ E)$ is the finest among all $\mathfrak{S}$-topologies on E' which are Schwartz topologies.*

Besides $\tau_{c_0}(E', E)$, we consider on E' the topology $\tau_n(E', E)$ of uniform convergence on all null sequences of E. While $\tau_{c_0}(E', E)$ only depends on the duality $\langle E, E'\rangle$, $\tau_n(E', E)$ depends even on the topology of E. Clearly, we have $\tau_{c_0}(E', E) \leqq \tau_n(E', E)$ in every case. Our next result deals with the question under which assumptions equality holds.

PROPOSITION 9. *The following statements are equivalent:*

(i) *E satisfies Mackey's convergence condition.*

(ii) *$\tau_n(E', E)$ is a Schwartz topology.*

PROOF. (i) $\Rightarrow$ (ii) is clear from Proposition 8. To prove (ii) $\Rightarrow$ (i), let $\tau_n(E', E)$ be a Schwartz topology. Since $\tau_n(E', E) \geqq \tau_{c_0}(E', E)$ is an $\mathfrak{S}$-topology, we have $\tau_n(E', E) = \tau_{c_0}(E, E)$ by Proposition 8; and every null sequence $(y_n)_{n\in\mathbb{N}}$ in E is contained in the closed absolutely convex hull M of some Mackey null sequence $(x_n)_{n\in\mathbb{N}}$ in E. As in the proof of Proposition 5, one shows that M is also the closed absolutely convex cover of $(x_n)_{n\in\mathbb{N}}$ in some E_B where B is a suitably chosen disk in E. Since on M the topologies induced from E_B and E coincide, $(y_n)_{n\in\mathbb{N}}$ is a null sequence in E_B and thus a Mackey null sequence in E.

4. Applications to bornological spaces

If E is a bornological space, then $\tau_{c_0}(E', E)$ is complete by [5, Sec. 28.5 (1)]. Since it is a Schwartz topology by Proposition 8, $(E', \tau_{c_0}(E', E))$ is a semi-reflexive space and its dual $\hat{E}$ is ultrabornological with respect to the topology $\tau(\hat{E}, E) = \beta(\hat{E}, E')$; refer to [2]. Using the Grothendieck construction of the completion $\tilde{E}$ of locally convex space E (refer to [5]), we may identify $\hat{E}$ with a linear subspace of $\tilde{E}$. Since the completion of a locally convex space with Mackay topology has its Mackey topology, too, E is a dense, linear, topological subspace of $\hat{E}$ under the Mackey topology $\tau(\hat{E}\ E')$. We have just proved the first and easy part of Theorem 10.

THEOREM 10. *Every bornological space E is isomorphic to a dense subspace of a locally convex space $\hat{E}$ having the following properties*:

(i) *$\hat{E}$ is ultrabornological,*

(ii) *$\hat{E}$ has a representation of the form $\hat{E} = \lim\operatorname{ind} \hat{E}_a, a \in A$, of separable Banach spaces $\hat{E}_a$,*

(iii) *every Mackey null sequence in E is* (*in the canonical way*) *a null sequence in some $\hat{E}_a$.*

Here we denote by lim ind the locally convex inductive limit (injective linking mappings).

PROOF. We have to prove (ii) and (iii). To reach our goal we use the fact proved recently in [4] and independently in [6] that the Schwartz space $(E', \tau_{c_0}(E', E))$ is isomorphic to a subspace of some topological Cartesian product $(c_0, \tau_c(c_0, l_1))^I$. Here I is a certain set depending on the cardinality of a suitably chosen neighborhood base of zero in $(E', \tau_{c_0}(E', E))$. By τ_c we denote the topology of uniform convergence on the compact subsets of the respective dual. Making use of the Grothendieck construction of the completion of a locally convex space, the completion of $(c_0, \tau_c(c_0, l_1))$ is easily seen to be isomorphic to $(l_\infty, \tau_c(l_\infty, l_1))$. By [5, Sec. 22.4, (3)], $\tau_c(l_\infty, l_1)$ is just the Mackey topology $\tau(l_\infty, l_1)$. Therefore we may and do identify $(E', \tau_{c_0}(E', E))$ with a closed (!) linear subspace of $(l_\infty, \tau(l_\infty, l_1))^I = (l_\infty^I, \tau(l_\infty^I, l_1^{(I)})$. By $l_1^{(I)}$ we mean the (locally convex) direct sum $\oplus_I l_1$.

From well-known theorems in the theory of dual pairings (refer to [2] or [5]) it follows that $\hat{E}$, under its Mackey topology $\tau(\hat{E}\ E')$, is linearly homeomorphic with the factor space $l_1^{(I)}/Q$, $Q = (E')^0$, supplied with its quotient topology. Hence we may identify $\hat{E}$ and $l_1\ /Q$ as topological vector spaces.

If we denote by A the set of all finite subsets of I, directed by inclusion, then we may write $l_1^{(I)} = \lim\operatorname{ind}\ l_1^a$ for $a \in A$, with $l_1^a = \prod_a l_1$. For every $a \in A$ put $Q_a = Q \cap l_1^a$ and $\hat{E}_a^\alpha = l_1^a/Q_a$. Then $\hat{E}_a$ is a separable Banach space and a straightforward computation shows that lim ind $\hat{E}_a$, $a \in A$, exists and is isomorphic to $\hat{E}$. So we have proved (ii).

Finally to obtain (iii), we first check that $\tau(l_\infty^I, l_1^{(I)})$ and $\tau_{c_0}(l_\infty^I, l_1^{(I)})$ coincide. Since $l_1^{(I)}$ is complete, $\tau_{c_0}(l_\infty^I, l_1^{(I)})$ is coarser than $\tau(l_\infty^I, l_1^{(I)})$. Since $\tau(l_\infty^I, l_1^{(I)})$ is a Schwartz topology and an $\mathfrak{S}$-topology, equality follows from Proposition 8.

Now let $(x_n)_{n\in\mathbb{N}}$ be any Mackey null sequence in E. Then there is a Mackey null sequence $(y_n)_{n\in\mathbb{N}}$ in $l_1^{(I)}$ such that $\{Ky_n \mid n \in \mathbb{N}\}^0$ is contained in $\{x_n \mid n \in \mathbb{N}\}^0$. Here $K: l_1^{(I)} \to \hat{E}$ denotes the canonical homomorphism, and the absolute polars are taken with respect to $\langle \hat{E}, E' \rangle$. But $(y_n)_{n\in\mathbb{N}}$ is a null sequence in some l_1^a, $a \in A$, hence $(Ky_n)_{n\in\mathbb{N}}$ is a null sequence in $\hat{E}_a$. The closed absolutely convex cover of $\{Ky_n \mid n \in \mathbb{N}\}$ in $\hat{E}_a$ is compact and therefore closed also in $\hat{E}$. The topologies induced on it by $\hat{E}$ and $\hat{E}_a$ are the same, hence $(x_n)_{n\in\mathbb{N}}$ is a null sequence in $\hat{E}_a$.

If E is a metrizable locally convex space, then we have $\hat{E} = \tilde{E}$ by the Banach-Dieudonné theorem.

It is easy to see that for every *locally complete*, locally convex space E the topology $\tau_{c_0}(E', E)$ is consistent with the duality $\langle E', E \rangle$. We may state as a corollary:

COROLLARY 11. *Every locally complete bornological space E is representable as the inductive limit $E = \operatorname{lim\,ind} E_a$, $a \in A$, of separable Banach spaces E_a such that each Mackey null sequence in E is a null sequence in some E_a.*

Of course, every locally complete bornological space is ultrabornological. Our last result gives necessary and sufficient conditions for such a space to satisfy Mackey's convergence condition. Let us agree to call a linear map T from a locally convex space E into a locally convex space F *sequentially invertible* if for every null sequence $(y_n)_{n\in\mathbb{N}}$ in $T(E)$ there exists a null sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $Tx_n = y_n$ holds for every n in $\mathbb{N}$.

PROPOSITION 12. *For every locally complete, locally convex space E the following statements are equivalent:*

(i) *E is bornological and satisfies Mackey's convergence condition;*

(ii) *E is isomorphic to some quotient $l_1^{(I)}/Q$ such that the quotient map $K: l_1^{(I)} \to l_1^{(I)}/Q$ is sequentially invertible;*

(iii) *E has a representation as the inductive limit $E = \operatorname{lim\,ind} E_a$, $a \in A$, of separable Banach spaces E_a such that every null sequence in E is a null sequence in some E_a.*

PROOF. We take up the construction given in the proof of Theorem (10) and the terminology introduced there.

(i) $\Rightarrow$ (ii): we identify $E = \hat{E}$ and $l_1^{(I)}/Q$. If $(x_n)_{n\in\mathbb{N}}$ is a null sequence in E we find a null sequence $(y_n)_{n\in\mathbb{N}}$ in some l_1^a, $a \in A$, such that the closed absolutely convex cover M of $\{Ky_n | n \in \mathbb{N}\}$ contains all the x_n. Since M is compact, $(x_n)_{n\in\mathbb{N}}$ is a null sequence in $E_a = l_1^a/Q_a$ and is therefore of the form $(Kt_n)_{n\in\mathbb{N}}$ where $(t_n)_{n\in\mathbb{N}}$ is a null sequence in the Banach space l_1^a, see [5, Sec. 22.2]. (ii) $\Rightarrow$ (iii) follows in the same way while (iii) $\Rightarrow$ (i) is obvious.

5. Fast convergence and ultrabornological spaces

In [8] and [9] M. de Wilde introduced the concept of fast convergence. A sequence $(x_n)_{n\in\mathbb{N}}$ in a locally convex space E is called a *fast null sequence* if it is a null sequence in some E_K, $K \subset E$ absolutely convex and compact. Obviously, the fast null sequences in E are the $\mathfrak{S}$-null sequences for the saturated hull $\mathfrak{S}'$ of the system of all absolutely convex and compact subsets in E. Thus the topology $\tau_{cf}(E', E)$ on E' of uniform convergence on the fast null sequences of E is the $\mathfrak{S}_0$-topology associated with $\mathfrak{S}$. Hence we can state the following.

PROPOSITION 13. *$\tau_{cf}(E', E)$ is a Schwartz topology on E' and consistent with the duality $\langle E', E\rangle$.*

Fast convergence is connected with ultrabornological spaces in the same way as Mackey convergence is connected with bornological spaces. Especially, a locally convex space E is ultrabornological if and only if its topology is the Mackey topology and $\tau_{cf}(E', E)$ is complete. A proof of this will be contained in the forthcoming second volume of [5]. Using this and the fact that for every set I even $\tau(l_\infty^I, l_1^{(I)})$ and $\tau_{cf}(l_\infty^I, l_1^{(I)})$ coincide, we obtain Proposition 14 exactly as we obtained Proposition 10.

PROPOSITION 14. *Every ultrabornological space E is representable as an inductive limit $E = \lim\operatorname{ind} E_a$, $a \in A$, of separable Banach spaces E_a such that every fast null sequence in E is a null sequence in some E_a.*

Finally, we answer the question of characterising those locally convex spaces E which satisfy the fast convergence condition, i.e., every null sequence in E is a fast null sequence.

We start with ultrabornological spaces by stating Proposition 15.

PROPOSITION 15. *For every locally convex space E the following statements are equivalent*:

(i) *E is ultrabornological and satisfies the fast convergence condition*;

(ii) *E is representable as an inductive limit $E = \lim\operatorname{ind} E_a$, $a \in A$, of separable Banach spaces such that every null sequence in E is a null sequence in some E_a.*

PROOF. (i) $\Rightarrow$ (ii) is a consequence of Proposition 14. If conversely (ii) holds, then E is ultrabornological and every null sequence in E is a fast null sequence E since every Banach space satisfies the fast convergence condition.

From Corollary 11 and Proposition 14 and from Propositions 12 and 15 we obtain the following two corollaries.

COROLLARY 16. *In every locally complete bornological space the fast null sequences and the Mackey null sequences coincide.*

COROLLARY 17. *A locally complete bornological space satisfies Mackey's convergence condition if and only if it satisfies the fast convergence condition.*

In the general case we have the following analogue to Proposition 9.

PROPOSITION 18. *A locally convex space E satisfies the fast convergence condition if and only if $\tau_n(E',E)$ is a Schwartz topology and the closed, absolutely convex cover of every null sequence in E is compact in E.*

PROOF. If E satisfies the fast convergence condition, then we have $\tau_n(E',E) = \tau_{c_f}(E',E)$. Thus our condition is necessary. To prove the converse, let $\tau_n(E',E)$ be a Schwartz topology and let the closed absolutely convex cover of every null sequence in E be compact. Then we have $\tau_n(E', E) \leqq \mathfrak{J}_{\mathfrak{S}}$ where $\mathfrak{S}$ is the saturated hull of the system of all absolutely convex compact sets in E. From this we get $\tau_n(E',E) = \tau_{c_f}(E',E)$ by Proposition 7. Now we proceed as in the proof of Proposition 9.

From Propositions 9 and 18 it follows that a quasi-complete locally convex space satisfies the Mackey convergence condition if and only if it satisfies the fast convergence condition. We do not intend to go into further details.

More results on these and related topics are contained in Johan Swart's doctoral dissertation, written under the supervision of Hans Jarchow at the University of Zurich.

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bornological space is representable as an inductive limit of separable Hilbert spaces such that the (injective) linking mappings are nuclear. Implicitly, H. Hogbé-Nlend has proved this result in the same way in [1].

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